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Quantum R matrix for G_2 and a solvable 175-vertex model

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Abstract. The quantum R matrix for Cartan's exceptional simple Lie algebra G_2 in its seven-dimensional (minimal) representation is presented. The R matrix is determined through the irreducible decomposition of the tensor modules over the quantised universal enveloping algebra $U_q(G_2)$. This defines a new solvable seven-state 175-vertex model on the planar square lattice whose Boltzmann weights satisfy the Yang-Baxter equation. For the equivalent face model, the local state probability is obtained in terms of a relative of a level-1 string function related to the affine Lie algebra pair $B_3^{(1)} \supset G_2^{(1)}$.

1. Introduction

Quantum group structures are playing important roles in many branches of mathematical physics. They have been a key to understanding the intimate relation among the recent developments in conformal field theories, operator algebras, link invariants and solvable lattice models. Study of the Yang-Baxter equation (YBE) is one origin of the basic notion of quantum groups, i.e. deformations of the universal enveloping algebras [1]. Given a simple Lie algebra g, the quantum group corresponds to a specific one-parameter deformation $U_q(g)$ of the universal enveloping algebra maintaining the Hopf algebra structure. The representation theory provides a systematic method for building the quantum R matrices, leading to a large family of trigonometric solutions of the YBE. A class of the quantum R matrices has been constructed along this line for Lie algebras of classical type; $sl(n)$, $so(n)$ and $sp(2n)$ [2, 3].

The purpose of this paper is to study an exceptional case. Namely, we determine the explicit form of the quantum R matrix for Cartan's exceptional simple Lie algebra G_2 in the fundamental seven-dimensional representation. As a result, a new solvable vertex model on the two-dimensional square lattice is obtained. The physical degree of freedom on each bond assumes seven possible values, corresponding to the weights of the representation. There is a constraint that the total weight of the left and the lower bonds of a vertex is equal to that of the right and the upper bonds. This yields 175 possible configurations round a vertex. To each of them a trigonometric function of the spectral parameter is assigned as the Boltzmann weight so as to satisfy the YBE.

This paper is also concerned with an analysis of the physical properties of the model. Following the line of the argument in [4], one can formulate the vertex model as a face model on the dual lattice. An interesting problem is then to evaluate the one-point functions called local state probabilities (LSP). We shall work this out by using Baxter's corner transfer matrix (CTM) method [5]. A series of works [6-9] has shown that the LSP calculation is intimately related to the representation theory of affine Lie algebras. Our treatment here is the first one for exceptional Lie algebras. We find that the LSP is described by an embedding $G_2^{(1)} \subset B_3^{(1)}$. Namely, the essential part of the result is a level-1 string function [10] of $G_2^{(1)}$ on a $B_3^{(1)}$ module viewed as a $G_2^{(1)}$ module through the embedding. We remark that the applicability of the CTM method is indeed due to a drastic simplification of the representation of $U_q(G_2)$ at $q=0$. This seems to provide a further insight into the quantum group, as well as the connection to the affine Lie algebras. Actually, such a phenomenon has been studied for $U_q(\mathrm{gl}(n, \boldsymbol{C}))$ [11], where a relation was shown with the Robinson–Shensted correspondence.

The organisation of the paper is as follows. In the next section we introduce the basic ingredients for characterising the quantum R matrix, e.g. the q-deformed universal enveloping algebra $U_q(G_2)$, the fundamental representation and the irreducible decomposition of the tensor product. We give the R matrix in the form of spectral decomposition, which involves orthogonal projectors onto the irreducible components of the decomposition. In section 3 we present a list of the q-Wigner coefficients constituting the projectors. We also note some of their properties which will be utilised in subsequent sections. Section 4 is devoted to the analysis of the resulting solvable lattice model in statistical mechanics. Using the relevant result for a $B_3^{(1)}$ vertex model in [4], we explicitly determine the LSP in terms of string functions. In order to make the descriptions self-contained, we include two appendices. Appendix 1 is a brief exposition of some of the results in [4] regarding the $B_n^{(1)}$ vertex model. Appendix 2 describes the embedding $G_2^{(1)} \subset B_3^{(1)}$.

We note that the R matrix $R(u, q)$ (u = spectral parameter) here extends those described earlier in special cases; [12] ($q=1$) and [13] ($|u| \to \infty$). The eigenvalues of the row-to-row transfer matrix have been discussed in [14].

Throughout the paper we assume that q is generic, i.e., $q \neq 0$ and $q^n \neq 1, \forall n \in \mathbb{Z}$ and use the notation

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad \varphi(x) = \prod_{j=1}^{\infty} (1 - x^j) \qquad E(z, x) = \prod_{j=1}^{\infty} (1 - zx^{j-1})(1 - z^{-1}x^j)(1 - x^j).$$

2. The quantum R matrix

2.1. The algebra $U_q(G_2)$

Let us begin by recalling basic facts about G_2 and fixing some notation. The algebra G_2 is a finite-dimensional simple Lie algebra associated with the Cartan matrix $(A_{ij})_{1 \leq i,j \leq 2}$, $A_{11} = A_{22} = 2$, $A_{12} = -3$, $A_{21} = -1$. The root system is well described by drawing two equilateral triangles rotated from each other by 60 degrees around the common centre (see figure 1). The four vectors pointing from the centre to the specified points correspond to the simple roots α_1, α_2 and the fundamental weights $\bar{\Lambda}_1, \bar{\Lambda}_2$. They are related as follows:

$$\bar{\Lambda}_1 = 2\alpha_1 + 3\alpha_2 \qquad \bar{\Lambda}_2 = \alpha_1 + 2\alpha_2. \tag{2.1}$$

We adopt the convention so that $|\text{long root}|^2 = 2$. Thus, for example, the inner products $\langle\,,\rangle$ among the roots have the following values: $\langle \alpha_1, \alpha_1 \rangle = 3\langle \alpha_2, \alpha_2 \rangle = 2, \langle \alpha_1, \alpha_2 \rangle = -1$.

For later convenience, we also introduce some notation for the affine Lie algebra $G_2^{(1)}$ [15]. Let Λ_0, Λ_1 and Λ_2 be the fundamental weights and δ the null root. We set $\rho = \Lambda_0 + \Lambda_1 + \Lambda_2$. In terms of the inner products among the classical parts $\bar{\Lambda}_i$ ($i = 1, 2$),

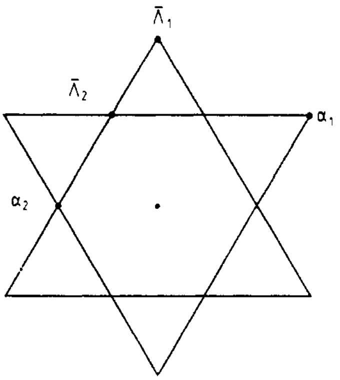

Figure 1. The simple roots α_1, α_2 and the fundamental weights $\bar{\Lambda}_1, \bar{\Lambda}_2$.

we define those on the space $\mathscr{H}^* = \mathbb{C}\Lambda_0 \oplus \mathbb{C}\Lambda_1 \oplus \mathbb{C}\Lambda_2 \oplus \mathbb{C}\delta$ by

$$\begin{gathered} \langle \Lambda_i, \Lambda_j \rangle = \langle \bar{\Lambda}_i, \bar{\Lambda}_j \rangle \qquad \bar{\Lambda}_i = \Lambda_i - \langle \delta, \Lambda_i \rangle \Lambda_0 \\ \langle \delta, \delta \rangle = 0 \qquad \langle \delta, \Lambda_0 \rangle = \langle \delta, \Lambda_2 \rangle = 1 \qquad \langle \delta, \Lambda_1 \rangle = 2. \end{gathered} \tag{2.2}$$

The quantity $\langle a, \delta \rangle$, $a \in \mathscr{H}^*$ is called the *level* of a. Note that $\langle \rho, \delta \rangle = 4$ is the dual Coxeter number of the algebra G_2.

Now we proceed to the description of the algebra $U_q = U_q(G_2)$ and its representations. Consider an associative algebra generated by the elements $X_i^\pm$, H_i $(i = 1, 2)$ under the relations:

$$\begin{gathered} [H_i, H_j] = 0 \qquad [H_i, X_j^\pm] = \pm A_{ji} X_j^\pm \qquad [X_1^\pm, X_2^\mp] = 0 \\ [X_1^+, X_1^-] = \frac{q^{3H_1} - q^{-3H_1}}{q^3 - q^{-3}} \qquad [X_2^+, X_2^-] = \frac{q^{H_2} - q^{-H_2}}{q - q^{-1}} \\ (X_1^\pm)^2 X_2^\pm - \frac{[6]}{[3]} X_1^\pm X_2^\pm X_1^\pm + X_2^\pm (X_1^\pm)^2 = 0 \\ (X_2^\pm)^4 X_1^\pm - [4](X_2^\pm)^3 X_1^\pm X_2^\pm + \frac{[3][4]}{[2]} (X_2^\pm)^2 X_1^\pm (X_2^\pm)^2 - [4] X_2^\pm X_1^\pm (X_2^\pm)^3 + X_1^\pm (X_2^\pm)^4 = 0. \end{gathered} \tag{2.3}$$

Our U_q is a q-analogue of the universal enveloping algebra of G_2 having the above commutation relations and endowed with the Hopf algebra structure [16]. A characteristic feature is the existence of the algebra homomorphism Δ, called comultiplication:

$$\begin{aligned} &\Delta : U_q \to U_q \otimes U_q \\ &\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \qquad i = 1, 2 \\ &\Delta(X_1^\pm) = X_1^\pm \otimes q^{-(3/2)H_1} + q^{(3/2)H_1} \otimes X_1^\pm \\ &\Delta(X_2^\pm) = X_2^\pm \otimes q^{-(1/2)H_2} + q^{(1/2)H_2} \otimes X_2^\pm. \end{aligned} \tag{2.4}$$

It is known [17] for generic values of q that every finite-dimensional representation of the simple Lie algebras naturally carries over to the one for their q-analogue defined as above. In view of this, we shall write V_Λ to represent the irreducible U_q module generated from the highest-weight vector $v(\Lambda)$, where $\Lambda \in \{\lambda_1 \bar{\Lambda}_1 + \lambda_2 \bar{\Lambda}_2 | \lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 0}\}$ and $v(\Lambda)$ defined (up to a constant multiple) by

$$X_i^+ v(\Lambda) = 0 \qquad H_i v(\Lambda) = \frac{2\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} v(\Lambda) \qquad \text{for } i = 1, 2. \tag{2.5}$$

The seven-dimensional representation mentioned in the previous section is realised on the highest-weight module $V_{\bar{\Lambda}_2}$. The weights contained therein are all multiplicity free. We name them ε_μ $(-3 \leqslant \mu \leqslant 3)$ as follows.

$$\begin{aligned} &\varepsilon_{-3} = \bar{\Lambda}_2 = \alpha_1 + 2\alpha_2 \qquad \varepsilon_{-2} = \alpha_1 + \alpha_2 \qquad \varepsilon_{-1} = \alpha_2 \\ &\varepsilon_\mu = -\varepsilon_{-\mu} \qquad -3 \leqslant \mu \leqslant 3. \end{aligned} \tag{2.6}$$

Note the obvious relation $\varepsilon_3 = \varepsilon_1 + \varepsilon_2$. In the rest of the paper, we shall fix the normalised weight vectors v_μ to have the weight ε_μ. Let $E_{\mu\nu}$ $(-3 \leqslant \mu, \nu \leqslant 3)$ be a matrix unit in $\mathrm{End}(V_{\bar{\Lambda}_2})$, i.e. $E_{\mu\nu} v_\lambda = \delta_{\nu\lambda} v_\mu$. Then the representation matrices $\pi(k)$ of the generators k are

$$\begin{aligned} &\pi(X_1^+) = {}^{\mathrm{t}}\pi(X_1^-) = E_{12} + E_{-2-1} \\ &\pi(X_2^+) = {}^{\mathrm{t}}\pi(X_2^-) = E_{23} + rE_{01} + rE_{-10} + E_{-3-2} \\ &\pi(H_1) = -E_{22} + E_{11} - E_{-1-1} + E_{-2-2} \\ &\pi(H_2) = -E_{33} + E_{22} - 2E_{11} + 2E_{-1-1} - E_{-2-2} + E_{-3-3} \\ &r = \sqrt{[2]} \end{aligned} \tag{2.7}$$

where ${}^{\mathrm{t}}$ denotes the transpose. We will also need the representation of the root vectors $X_\theta^\pm$ and H_θ associated with the maximal root $\theta = \bar{\Lambda}_1$ ($V_{\bar{\Lambda}_1}$ is the adjoint representation). They are given by

$$\pi(X_\theta^+) = {}^{\mathrm{t}}\pi(X_\theta^-) = E_{-23} + E_{-32} \qquad \pi(H_\theta) = E_{-2-2} + E_{-3-3} - E_{33} - E_{22}. \tag{2.8}$$

2.2. *Characterisation of the quantum R matrix*

Let $R(u) = R(u, q)$ be an element of $\mathrm{End}(V_{\bar{\Lambda}_2} \otimes V_{\bar{\Lambda}_2})$, where $u \in \mathbb{C}$ is a spectral parameter. The quantum R matrix corresponding to the pair $(\mathrm{G}_2^{(1)}, \pi)$ is the solution of the following equations in $\mathrm{End}(V_{\bar{\Lambda}_2} \otimes V_{\bar{\Lambda}_2})$:

$$[R(u), \Delta(\mathrm{U}_q)] = 0 \tag{2.9a}$$

$$\begin{aligned} &R(u)(X_\theta^- \otimes q^{(3/2)H_\theta} + q^{-2u} q^{-(3/2)H_\theta} \otimes X_\theta^-) \\ &\qquad = (q^{-2u} X_\theta^- \otimes q^{(3/2)H_\theta} + q^{-(3/2)H_\theta} \otimes X_\theta^-) R(u). \end{aligned} \tag{2.9b}$$

Up to a normalisation, these are known to characterise the matrix $R(u)$ uniquely due to Jimbo [3]. Moreover, one can show that the solution to the equations satisfy the YBE in $\mathrm{End}(V_{\bar{\Lambda}_2} \otimes V_{\bar{\Lambda}_2} \otimes V_{\bar{\Lambda}_2})$ (compare with [3])

$$(R(u) \otimes 1)(1 \otimes R(u+v))(R(v) \otimes 1) = (1 \otimes R(v))(R(u+v) \otimes 1)(1 \otimes R(u)). \tag{2.10}$$

To solve (2.9), we first decompose the $\mathrm{U}_q \otimes \mathrm{U}_q$ module $V_{\bar{\Lambda}_2} \otimes V_{\bar{\Lambda}_2}$ into irreducible modules with respect to $\Delta(\mathrm{U}_q)$:

$$V_{\bar{\Lambda}_2} \otimes V_{\bar{\Lambda}_2} = V_{2\bar{\Lambda}_2} \oplus V_{\bar{\Lambda}_1} \oplus V_{\bar{\Lambda}_2} \oplus V_0 \tag{2.11}$$

which is parallel with the classical case $q = 1$. Since $R(u)$ belongs to the commutant of $\Delta(\mathrm{U}_q)$ (2.9a), it must be of the form

$$R(u) = \sum_{\Lambda = 2\bar{\Lambda}_2, \bar{\Lambda}_1, \bar{\Lambda}_2, 0} \rho_\Lambda(u) \mathscr{P}_\Lambda. \tag{2.12a}$$

Here the $\mathscr{P}_\Lambda$ are the orthogonal projectors onto the irreducible pieces V_Λ and the eigenvalues $\rho_\Lambda(u)$ are the functions to be determined. The projectors will be described in the next section. In particular, relations among the operators $\mathscr{P}_\Lambda X_\theta^- \otimes q^{(3/2)H_\theta}\mathscr{P}_{\Lambda'}$ and $\mathscr{P}_\Lambda q^{-(3/2)H_\theta} \otimes X_\theta^- \mathscr{P}_{\Lambda'}$ are determined as in (3.6). Substituting (2.12*a*) into (2.9*b*) and using (3.6), we get

$$[4][6]\rho_\Lambda(u) = \begin{cases} [1+u][4+u][6+u] & \text{for } \Lambda = 2\bar{\Lambda}_2 \\ [1-u][4+u][6+u] & \text{for } \Lambda = \bar{\Lambda}_1 \\ [1+u][4-u][6+u] & \text{for } \Lambda = \bar{\Lambda}_2 \\ [1-u][4+u][6-u] & \text{for } \Lambda = 0. \end{cases} \tag{2.12b}$$

Here we have chosen a normalisation such that $R(0)$ is the identity. The spectral decomposition (2.12) recovers Reshetikhin's formula [13] in the limit $q^{\pm u} \to \infty$:

$$\lim_{q^{\pm u} \to \infty} [4][6](q-q^{-1})^3 q^{\pm(3u+9)} R(u) = \pm \sum_\Lambda \varepsilon(\Lambda) q^{\pm(3/2)(c(\Lambda)-2c(\Lambda_2))} \mathscr{P}_\Lambda \tag{2.13a}$$

$$c(\Lambda) = \langle \Lambda, \Lambda + 2\rho \rangle \tag{2.13b}$$

$$\varepsilon(2\bar{\Lambda}_2) = \varepsilon(0) = 1 \qquad \varepsilon(\bar{\Lambda}_1) = \varepsilon(\bar{\Lambda}_2) = -1. \tag{2.13c}$$

The eigenvalues of the constant R matrix (braid operator) are thus readily obtained in terms of the Casimir values $c(\Lambda)$.

3. The projectors and the *q*-Wigner coefficients

Here we explicitly construct the projectors $\mathscr{P}_\Lambda$ appearing in (2.12*a*) and exhibit the resulting properties of the R matrix. A similar description can be found in [13]. Consider the irreducible decomposition (2.11). We denote by $\{v_i^{(\Lambda)} | 1 \leq i \leq \dim V_\Lambda\}$ a set of orthonormal weight vectors of V_Λ. They are expressed in the form

$$v_i^{(\Lambda)} = \sum_{\mu\nu} C_{i\mu\nu}^{(\Lambda)}(q) v_\mu \otimes v_\nu \tag{3.1}$$

where the q-Wigner coefficient $C_{i\mu\nu}^{(\Lambda)}(q)$ is assumed to be zero unless $\varepsilon_\mu + \varepsilon_\nu =$ the weight of $v_i^{(\Lambda)}$. (Note that the index i does *not* directly correspond to the weight.) From the requirement that the $v_i^{(\Lambda)}$ and the v_μ are both orthonormal, we readily see that

$$\sum_{\mu\nu} C_{i\mu\nu}^{(\Lambda)*}(q) C_{j\mu\nu}^{(\Lambda')}(q) = \delta_{ij}\delta_{\Lambda\Lambda'} \tag{3.2}$$

where * stands for complex conjugation. The projector $\mathscr{P}_\Lambda$ is then given by

$$\mathscr{P}_\Lambda = \sum_{i=1}^{\dim V_\Lambda} \sum_{\mu\nu\kappa\lambda} C_{i\mu\nu}^{(\Lambda)*}(q) C_{i\kappa\lambda}^{(\Lambda)}(q) E_{\mu\kappa} \otimes E_{\nu\lambda} \tag{3.3}$$

in which $\mathscr{P}_\Lambda \mathscr{P}_{\Lambda'} = \delta_{\Lambda\Lambda'} \mathscr{P}_\Lambda$ is evident because of (3.2). Combining this with (2.12*a*), we get

$$R(u,q) = \sum_{\mu\nu\kappa\lambda} R(u,q)_{\mu\nu\kappa\lambda} E_{\mu\kappa} \otimes E_{\nu\lambda} \tag{3.4a}$$

where the matrix element is

$$R(u,q)_{\mu\nu\kappa\lambda} = R(u,q)_{\kappa\lambda\mu\nu} = \sum_\Lambda \rho_\Lambda(u) \sum_{i=1}^{\dim V_\Lambda} C_{i\mu\nu}^{(\Lambda)*}(q) C_{i\kappa\lambda}^{(\Lambda)}(q). \tag{3.4b}$$

In fact, if $v_i^{(\Lambda)}$ in (3.1) is a weight vector having the weight $\varepsilon_\mu + \varepsilon_\nu$, then so is the vector $\Sigma_{\mu\nu} C_{i\mu\nu}^{(\Lambda)}(q) v_{-\nu} \otimes v_{-\mu}$ with the negated weight. Thus we present the orthonormal vectors (3.1) corresponding to the non-positive weights. In the case that the dimensionality of the weight space exceeds 1, there is an arbitrariness in choosing the weight vectors, which does not affect the projector (3.3). The weight vectors for $V_{2\bar{\Lambda}_2}$ (dimension = 27) are as follows:

$$v_1^{(2\bar{\Lambda}_2)} = v_3 \otimes v_3 \qquad v_2^{(2\bar{\Lambda}_2)} = r^{-1}(q^{1/2} v_2 \otimes v_3 + q^{-1/2} v_3 \otimes v_2)$$

$$v_3^{(2\bar{\Lambda}_2)} = r^{-1}(q^{1/2} v_1 \otimes v_3 + q^{-1/2} v_3 \otimes v_1) \qquad v_4^{(2\bar{\Lambda}_2)} = v_2 \otimes v_2$$

$$v_5^{(2\bar{\Lambda}_2)} = \frac{1}{\sqrt{[3]}} [r^{-1}(q^{1/2} v_2 \otimes v_1 + q^{-1/2} v_1 \otimes v_2) + q v_0 \otimes v_3 + q^{-1} v_3 \otimes v_0]$$

$$v_6^{(2\bar{\Lambda}_2)} = \left(\frac{[4]}{[3][8]}\right)^{1/2} [r(q^{5/2} v_1 \otimes v_2 + q^{-5/2} v_2 \otimes v_1) - q v_0 \otimes v_3 - q^{-1} v_3 \otimes v_0]$$

$$v_7^{(2\bar{\Lambda}_2)} = v_1 \otimes v_1$$

$$v_8^{(2\bar{\Lambda}_2)} = \frac{1}{\sqrt{[3]}} [r^{-1}(q^{3/2} v_{-1} \otimes v_3 + q^{-3/2} v_3 \otimes v_{-1}) + v_0 \otimes v_2 + v_2 \otimes v_0]$$

$$v_9^{(2\bar{\Lambda}_2)} = \left(\frac{[4]}{[3][8]}\right)^{1/2} [(q^3 v_0 \otimes v_2 + q^{-3} v_2 \otimes v_0 - r(q^{3/2} v_{-1} \otimes v_3 + q^{-3/2} v_3 \otimes v_{-1})]$$

$$v_{10}^{(2\bar{\Lambda}_2)} = r^{-1}(q^{1/2} v_{-1} \otimes v_2 + q^{-1/2} v_2 \otimes v_{-1}) \tag{3.5a}$$

$$v_{11}^{(2\bar{\Lambda}_2)} = \left(\frac{[2]}{[4]}\right)^{1/2} (q v_0 \otimes v_1 + q^{-1} v_1 \otimes v_0)$$

$$v_{12}^{(2\bar{\Lambda}_2)} = \frac{1}{\sqrt{[8]}} \left(r(q - q^{-1})(q v_1 \otimes v_0 - q^{-1} v_0 \otimes v_1) + \frac{[4]}{[2]} (q^{3/2} v_{-2} \otimes v_3 + q^{-3/2} v_3 \otimes v_{-2}) \right)$$

$$v_{13}^{(2\bar{\Lambda}_2)} = \left(\frac{[2]}{[3][4]}\right)^{1/2} (q^2 v_{-1} \otimes v_1 + q^{-2} v_1 \otimes v_{-1} + [2] v_0 \otimes v_0)$$

$$v_{14}^{(2\bar{\Lambda}_2)} = \frac{1}{\sqrt{[2][8]}} \left(\frac{[4]}{[2]} (q^2 v_{-3} \otimes v_3 + q^{-2} v_3 \otimes v_{-3} + q v_{-2} \otimes v_2 + q^{-1} v_2 \otimes v_{-2}) \right.$$
$$\left. + (q - q^{-1})[2](v_1 \otimes v_{-1} - v_{-1} \otimes v_1) + (q - q^{-1})^2 [2] v_0 \otimes v_0 \right)$$

$$v_{15}^{(2\bar{\Lambda}_2)} = \frac{1}{[2]\sqrt{[3][7]}} [[3](-q^2 v_{-3} \otimes v_3 - q^{-2} v_3 \otimes v_{-3} + q^3 v_{-2} \otimes v_2 + q^{-3} v_2 \otimes v_{-2})$$
$$+ [4](q^{-1} v_{-1} \otimes v_1 + q v_1 \otimes v_{-1} - v_0 \otimes v_0)].$$

The weight vectors for $V_{\bar{\Lambda}_1}$ (dimension = 14) are as follows:

$$v_1^{(\bar{\Lambda}_1)} = r^{-1}(q^{-1/2} v_2 \otimes v_3 - q^{1/2} v_3 \otimes v_2)$$

$$v_2^{(\bar{\Lambda}_1)} = r^{-1}(q^{-1/2} v_1 \otimes v_3 - q^{1/2} v_3 \otimes v_1)$$

$$v_3^{(\bar{\Lambda}_1)} = \frac{1}{\sqrt{[2][3]}} [r(v_0 \otimes v_3 - v_3 \otimes v_0) + q^{-3/2} v_1 \otimes v_2 - q^{3/2} v_2 \otimes v_1]$$

$$v_4^{(\bar{\Lambda}_1)} = \frac{1}{\sqrt{[2][3]}} [q^{1/2} v_{-1} \otimes v_3 - q^{-1/2} v_3 \otimes v_{-1} + r(q^{-1} v_0 \otimes v_2 - q v_2 \otimes v_0)]$$

$$v_5^{(\bar\Lambda_1)}=\frac{1}{\sqrt{[2][3]}}[q^{1/2}v_{-2}\otimes v_3-q^{-1/2}v_3\otimes v_{-2}+r(q^{-1}v_0\otimes v_1-qv_1\otimes v_0)] \quad (3.5b)$$

$$v_6^{(\bar\Lambda_1)}=r^{-1}(q^{-1/2}v_{-1}\otimes v_2-q^{1/2}v_2\otimes v_{-1})$$

$$v_7^{(\bar\Lambda_1)}=\frac{1}{[2]\sqrt{[3]}}[qv_{-3}\otimes v_3-q^{-1}v_3\otimes v_{-3}+v_{-2}\otimes v_2-v_2\otimes v_{-2}$$
$$+[2](v_{-1}\otimes v_1-v_1\otimes v_{-1})-(q-q^{-1})[2]v_0\otimes v_0]$$

$$v_8^{(\bar\Lambda_1)}=\frac{1}{[2]}\left(\frac{[4][6]}{[2][12]}\right)^{1/2}[q^{-1}v_3\otimes v_{-3}-qv_{-3}\otimes v_3+q^2v_{-2}\otimes v_2-q^{-2}v_2\otimes v_{-2}$$
$$+(q-q^{-1})[2](v_0\otimes v_0-q^{-1}v_{-1}\otimes v_1-qv_1\otimes v_{-1})].$$

The weight vectors for $V_{\bar\Lambda_2}$ (dimension = 7) are as follows:

$$v_1^{(\bar\Lambda_2)}=\left(\frac{[4]}{[3][8]}\right)^{1/2}[r(q^{3/2}v_2\otimes v_1-q^{-3/2}v_1\otimes v_2)+q^{-3}v_0\otimes v_3-q^3v_3\otimes v_0]$$

$$v_2^{(\bar\Lambda_2)}=\left(\frac{[4]}{[3][8]}\right)^{1/2}[r(q^{-5/2}v_{-1}\otimes v_3-q^{5/2}v_3\otimes v_{-1})+qv_2\otimes v_0-q^{-1}v_0\otimes v_2]$$

$$v_3^{(\bar\Lambda_2)}=\left(\frac{[4]}{[3][8]}\right)^{1/2}[r(q^{-5/2}v_{-2}\otimes v_3-q^{5/2}v_3\otimes v_{-2})+qv_1\otimes v_0-q^{-1}v_0\otimes v_1] \quad (3.5c)$$

$$v_4^{(\bar\Lambda_2)}=\left(\frac{[4]}{[3][8]}\right)^{1/2}[q^{-2}v_{-3}\otimes v_3-q^2v_3\otimes v_{-3}+q^{-3}v_{-2}\otimes v_2-q^3v_2\otimes v_{-2}$$
$$+v_1\otimes v_{-1}-v_{-1}\otimes v_1+(q-q^{-1})v_0\otimes v_0].$$

The weight vector for V_0 (dimension = 1) is

$$v_1^{(0)}=\left(\frac{[4][6]}{[2][7][12]}\right)^{1/2}(q^5v_3\otimes v_{-3}+q^{-5}v_{-3}\otimes v_3-q^4v_2\otimes v_{-2}-q^{-4}v_{-2}\otimes v_2$$
$$+qv_1\otimes v_{-1}+q^{-1}v_{-1}\otimes v_1-v_0\otimes v_0). \quad (3.5d)$$

Using (2.8), (3.3) and (3.5), one can show the following relations:

$$\mathcal{P}_\Lambda X_\theta^-\otimes q^{(3/2)H_\theta}\mathcal{P}_\Lambda=\mathcal{P}_\Lambda q^{-(3/2)H_\theta}\otimes X_\theta^-\mathcal{P}_\Lambda\neq 0 \qquad \text{if } \Lambda\in\{2\bar\Lambda_2,\bar\Lambda_1,\bar\Lambda_2\}$$
$$\mathcal{P}_\Lambda X_\theta^-\otimes q^{(3/2)H_\theta}\mathcal{P}_{\Lambda'}=-\mathcal{P}_\Lambda q^{-(3/2)H_\theta}\otimes X_\theta^-\mathcal{P}_{\Lambda'}q^{-(3/2)(c(\Lambda)-c(\Lambda'))}\neq 0 \quad (3.6)$$
$$\text{if } (\Lambda,\Lambda') \text{ or } (\Lambda',\Lambda)\in\{(\bar\Lambda_1,2\bar\Lambda_2),(\bar\Lambda_2,2\bar\Lambda_2),(0,\bar\Lambda_1)\}.$$

All the other combinations $\mathcal{P}_\Lambda X_\theta^-\otimes q^{(3/2)H_\theta}\mathcal{P}_{\Lambda'}$ and $\mathcal{P}_\Lambda q^{-(3/2)H_\theta}\otimes X_\theta^-\mathcal{P}_{\Lambda'}$ are zero. These relations, together with (2.9*b*) and (2.12*a*), yield the result (2.12*b*).

We now state some properties of the R matrix.

(i) $$R(u,q)_{\mu\nu\kappa\lambda}\neq 0 \qquad \text{iff } \varepsilon_\mu+\varepsilon_\nu=\varepsilon_\kappa+\varepsilon_\lambda \quad (3.7a)$$

$$R(u,q)_{\mu\nu\kappa\lambda}=R(u,q^{-1})_{\nu\mu\lambda\kappa}. \quad (3.7b)$$

The support property (3.7*a*) is immediately obvious if we note that $[R(u,q),\Delta(H_i)]=0$, $i=1,2$ in (2.9*a*). There are 175 possible choices of μ, ν, κ and λ satisfying this condition. On the other hand, $(3.7b)$ holds due to the property $C^{(\Lambda)}_{i\mu\nu}(q) = \varepsilon(\Lambda) C^{(\Lambda)}_{i\nu\mu}(q^{-1})$, where the sign factor $\varepsilon(\Lambda)$ has been defined in $(2.13c)$. By direct computation using $(3.4b)$ and (3.5), one can also verify the following.

$$\text{(ii)} \qquad R(-6-u, q)_{\mu\nu\kappa\lambda} = \left(\frac{g_\mu g_\kappa}{g_\nu g_\lambda}\right)^{1/2} R(u, q)_{-\kappa\mu\lambda-\nu} \qquad g_\mu = q^{-3(\varepsilon_\mu, \rho)} \tag{3.8}$$

$$\text{(iii)} \qquad \lim_{q\to 0, w=q^{-2u};\text{fix}} R(u, q) w^{-1/2} = \sum_{\mu\nu} w^{H(\varepsilon_\mu, \varepsilon_\nu)} E_{\mu\mu} \otimes E_{\nu\nu} \tag{3.9a}$$

where the function $H(\varepsilon_\mu, \varepsilon_\nu)$ is defined by

$$H(\varepsilon_\mu, \varepsilon_\nu) = \begin{cases} 0 & \text{if } \mu < \nu \\ 1 & \text{otherwise} \end{cases}$$

with the two exceptions

$$H(\varepsilon_{-3}, \varepsilon_3) = -1 \qquad H(\varepsilon_0, \varepsilon_0) = 0. \tag{3.9b}$$

In fact, (3.9) is due to the fact that each weight space of the V_Λ on the RHS of (2.11) is spanned by decomposable vectors (pure tensors) $v_\mu \otimes v_\nu$ in the limit $q \to 0$. These properties will be used in the evaluation of the local-state probability in the next section.

4. Related solvable lattice models and the local-state probability

4.1. A solvable vertex model and the face formulation

The quantum R matrix constructed in the previous sections gives rise to a solvable statistical model, which we are now going to study. Consider a two-dimensional square lattice $\mathscr{L}$ with a fluctuation variable $\alpha^{(i)}$ assigned to each bond i. We assume that $\alpha^{(i)}$ takes the values in the set $\{\varepsilon_\mu | -3 \leq \mu \leq 3\}$ of the weights in the fundamental representation. Let ε_μ, ε_ν, ε_κ and ε_λ be the weights on the left, lower, upper and right bond of a vertex, respectively. To such a configuration round a vertex, we attach the matrix element $R(u, q)_{\mu\nu\kappa\lambda}$ $(3.4b)$ as the Boltzmann weight. This yields a seven-state 175-vertex model whose Boltzmann weights obey the YBE. It can also be formulated as a face model following [4]. Let us consider the planar square lattice $\mathscr{L}'$ dual to the $\mathscr{L}$ on which the vertex model was defined. We put a site variable $\sigma^{(i)}$ on each site i of $\mathscr{L}'$ and let it range over the set I of the level-1 integral weights of the affine Lie algebra $G_2^{(1)}$, i.e.

$$I = \{a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2 | a_0, a_1, a_2 \in \mathbb{Z}, a_0 + 2a_1 + a_2 = 1\}. \tag{4.1}$$

We shall call an element of I a local state or simply state. For two states a and b, we define an ordered pair (a, b) to be admissible if and only if $b - a = \varepsilon_\mu$ for some $-3 \leq \mu \leq 3$. Because of (2.6), the admissibility of (a, b) and (b, a) is equivalent. Let a, b, c and d be the states on the NW, NE, SE and SW corners of a face of $\mathscr{L}'$. To such a state configuration, we assign a Boltzmann weight $W(a, b, c, d|u)$ via the rule

$$W(a, b, c, d|u) = \begin{cases} R(u, q)_{\mu\nu\kappa\lambda} & \text{if } b-a=\varepsilon_\kappa,\ c-b=\varepsilon_\lambda,\ d-a=\varepsilon_\mu,\ c-d=\varepsilon_\nu \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Thus every adjacent pair of the states must be admissible in the above sense. From the definition (4.2) and the YBE (2.10), it follows that the $W(a, b, c, d|u)$ obeys the star-triangle relation [5]. It is also easy to see the following properties of the Boltzmann weights:

$$W(a,b,c,d|u)=W(a,d,c,b|u) \quad (4.3a)$$
$$= W(a+r,b+r,c+r,d+r|u) \qquad \forall r\in I \quad (4.3b)$$
$$W(a,b,c,d|-6-u)=\left(\frac{G_bG_d}{G_aG_c}\right)^{1/2} W(b,c,d,a|u) \quad (4.3c)$$
$$\left.\begin{aligned}&G_a=q^{-6(a,\rho)}\\ &\lim_{q\to 0,w=q^{-2u};\text{fix}} W(a,b,c,d|u)w^{-1/2}=\delta_{bd}w^{H(b-a,c-b)}\end{aligned}\right\} \quad (4.3d)$$

where $H(b-a,c-b)$ is given by (3.9b). These are direct consequences of (3.4b), (3.8), (3.9) and (4.2).

Now we proceed to the evaluation of the local-state probability (LSP) in this face formulation. By definition, it is the probability of finding a site variable of $\mathscr{L}'$, say $\sigma^{(1)}$, in a preassigned state $a\in I$ under certain boundary conditions:

$$\left.\begin{aligned}&Z=\sum_{\text{configurations}}\prod_{\text{faces}} W(\sigma^{(i)},\sigma^{(j)},\sigma^{(k)},\sigma^{(l)}|u)\\ &P_{\sigma^{(1)}=a}=Z^{-1}\sum_{\text{configurations}}\delta_{\sigma^{(1)}a}\prod_{\text{faces}} W(\sigma^{(i)},\sigma^{(j)},\sigma^{(k)},\sigma^{(l)}|u).\end{aligned}\right\} \quad (4.4)$$

The star-triangle relation and the properties (4.3) imply that we can invoke the corner transfer matrix method [5]. In the following we shall deal exclusively with the case $0<q<1, |q^{-2u}|<1$. We fix the boundary condition to the *ground states*, i.e. those configurations giving the maximal contribution to Z. In order to specify it, let us consider the limit $q\to 0$ keeping $w=q^{-2u}$ fixed. From (4.3d) the system is then frozen to those configurations invariant under the translation along the SW–NE direction. These are labelled by a one-dimensional sequence of the states $\{\sigma^{(j)}|\sigma^{(j)}\in I, j\in \mathbb{Z}, (\sigma^{(j)},\sigma^{(j+1)})$; admissible$\}$ sitting on the horizontal line containing $\sigma^{(1)}$. Moreover, by the assumption $|w|<1$, there are essentially the three following ground states $\{\sigma_k^{(j)}\}_{j\in\mathbb{Z}}\ k=0,1,2$:

$$\left.\begin{aligned}&\sigma_k^{(j)}=\begin{cases}\Lambda_0 & \text{if } j\equiv k \bmod 2\\ \Lambda_2 & \text{otherwise}\end{cases} \qquad \text{for } k=0,1\\ &\sigma_2^{(j)}=\Lambda_2 \qquad \forall_j.\end{aligned}\right\} \quad (4.5)$$

Thus we consider the LSP $P(a|\sigma_k)$ $(k=0,1,2)$ under the condition that $\sigma^{(j)}=\sigma_k^{(j)}$ for $|j|\gg 1$. Applying the corner transfer matrix method, we deduce the following expression:

$$P(a|\sigma_k)=\lim_{m\to\infty} P_m(a|\sigma_k^{(m+1)},\sigma_k^{(m+2)}) \quad (4.6a)$$
$$P_m(a|b,c)=\frac{G_af_m(b-a,c-b;q^{24})}{\sum_{a'\in I}G_{a'}f_m(b-a',c-b;q^{24})}. \quad (4.6b)$$

Here the function $f_m(\gamma,\varepsilon_\nu;q)$ $(m\in\mathbb{Z}_{\geqslant 0}, \gamma\in\mathbb{Z}\varepsilon_1+\mathbb{Z}\varepsilon_2+\mathbb{Z}\varepsilon_3=\mathbb{Z}\varepsilon_1\oplus\mathbb{Z}\varepsilon_2)$ is the one-dimensional (1D) configuration sum:

$$f_m(\gamma,\varepsilon_\mu;q)=\sum q^{\sum_{j=1}^m jH(\gamma^{(j)},\gamma^{(j+1)})}. \quad (4.7a)$$

The outer sum extends over $\gamma^{(j)}\in\{\varepsilon_\nu|-3\leqslant\nu\leqslant 3\}$ $(1\leqslant j\leqslant m+1)$ under the constraint

$$\sum_{j=1}^m \gamma^{(j)}=\gamma \qquad \gamma^{(m+1)}=\varepsilon_\mu. \quad (4.7b)$$

4.2. *The one-dimensional sum and string functions*

The 1D sum (4.7) exhibits a connection of our model to the representation theory of affine Lie algebras. It turns out to be a relative of the level-1 string function [10] of $G_2^{(1)}$ on $B_3^{(1)}$ modules. In order to state this, we have summarised the necessary information about a $B_n^{(1)}$ vertex model in appendix 1. The principal result therein is that the 1D sum $g_m(\xi, e_i; q)$ (A1.3) yields the string functions of the level-1 $B_n^{(1)}$ modules $L(\omega_0)$, $L(\omega_1)$ and $L(\omega_n)$ as in (A1.6). On the other hand, for $n=3$ we can regard these modules as $G_2^{(1)}$ modules through the embedding ϕ^*: $G_2^{(1)} \hookrightarrow B_3^{(1)}$ described in appendix 2. The resulting $G_2^{(1)}$ modules are no longer irreducible and the string functions on them will become the linear superposition of (A1.6*a*) over the inverse image of the map ϕ (A2.1). This has been exactly realised in the following relation between $f_m(\gamma, \varepsilon_\mu; x)$ and $g_m(\xi, e_i; x)$:

$$f_m(\gamma, \varepsilon_\mu; x) = \sum_\xi g_m(\xi, e_i; x). \tag{4.8}$$

On the RHS, e_i is uniquely determined by $\phi(e_i) = \varepsilon_\mu$ (see (A2.2)) and the sum ranges over $\xi \in \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ such that $\phi(\xi) = \gamma$ ($\xi_1 + \xi_3 = \gamma_1$, $\xi_1 + \xi_2 = \gamma_2$ if we write $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$, $\gamma = \gamma_1 \varepsilon_{-1} + \gamma_2 \varepsilon_{-2}$). The equality (4.8) directly follows from the definitions and the fact that

$$H(\varepsilon_\mu, \varepsilon_\nu) = \hat{H}(e_i, e_j) \qquad \text{if } \phi(e_i) = \varepsilon_\mu,\ \phi(e_j) = \varepsilon_\nu \tag{4.9}$$

(see (3.9*b*), (A1.3*a*) and (A2.2)). On the other hand, the ground states (4.5) and (A1.5) are related to each other by

$$\phi(\hat{\sigma}_{\omega_{\hat{k}}}^{(j)}) = \sigma_k^{(j)} \qquad \forall_j \in \mathbb{Z} \tag{4.10}$$

where $\hat{k} = 0, 1$ and 3 for $k = 0, 1$ and 2, respectively. Combining this with (4.8) and (A1.6), we find that the numerator of (4.6*b*) is proportional to ($x = q^{24}$):

$$\lim_{m\to\infty} x^{-\psi_m(\omega_{\hat{k}})} f_m(\sigma_k^{(m+1)} - a, \sigma_k^{(m+2)} - \sigma_k^{(m+1)}; x) = \sum_{\hat{a}\in \hat{I}_a} \sum_i \dim L(\omega_{\hat{k}})_{\hat{a}-i\hat{\delta}} x^i. \tag{4.11}$$

Here $a \in I$ and the set $\hat{I}_a$ is defined by $\hat{I}_a = \{\hat{a} \in \mathbb{Z}\omega_0 \oplus \ldots \oplus \mathbb{Z}\omega_3 | \langle \hat{a}, \hat{\delta}\rangle = 1, \phi(\hat{a}) = a\}$. Thus we conclude that our 1D sum (in the limit $m \to \infty$) is the level-1 string function of $G_2^{(1)}$ on $B_3^{(1)}$ modules viewed as the $G_2^{(1)}$ modules through the embedding ϕ^*.

4.3. *The local-state probability*

Having established a characterisation of our 1D sum, we now turn to the LSP itself. By virtue of (4.11), the expression (4.6) is rewritten as follows:

$$P(a|\sigma_k) = \frac{\sum_{\hat{a}\in\hat{I}_a} \sum_i \dim L(\omega_{\hat{k}})_{\hat{a}-i\hat{\delta}} q^{-6\langle\phi(\hat{a}-i\hat{\delta}),\rho\rangle}}{\sum_{\hat{\mu}} \dim L(\omega_{\hat{k}})_{\hat{\mu}} q^{-6\langle\phi(\hat{\mu}),\rho\rangle}}. \tag{4.12}$$

In what follows we shall outline the calculation of the explicit form of $P(a = a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2|\sigma_0)$. The other cases can be dealt with quite similarly. Consider the quantity $f_m(\sigma_0^{(m+1)} - a, \sigma_0^{(m+2)} - \sigma_0^{(m+1)}; x)$ appearing in (4.11). For even m, this is equal to

$$\sum_{\xi_1+\xi_3 = 1-a_1-a_2,\ \xi_1+\xi_2 = 1-2a_1-a_2} g_m(\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3, e_{-1}; x) \tag{4.13}$$

where we have used (4.5), (4.8) and eliminated a_0 by (4.1). Taking (A1.7) into account with $n=3$, $i=-1$ and $\frac{1}{2}|\xi-\omega_1|^2-\frac{1}{2}=\frac{1}{2}[\xi_1(\xi_1-2)+\xi_2^2+\xi_3^2]$, we have the limit

$$\lim_{m\to\infty, m\in 2\mathbb{Z}} x^{m/2} f_m(\sigma_0^{(m+1)}-a, \sigma_0^{(m+2)}-\sigma_0^{(m+1)}; x)$$

$$=\frac{\varphi(-\sqrt{x})}{2\varphi(x)^4}\sum_{\xi_1\in\mathbb{Z}} x^Q+\frac{\varphi(\sqrt{x})}{2\varphi(x)^4}\sum_{\xi_1\in\mathbb{Z}}(-1)^{\xi_1-3a_1}x^Q \tag{4.14}$$

$$Q=\tfrac{1}{2}\xi_1(\xi_1-2)+\tfrac{1}{2}(\xi_1+2a_1+a_2-1)^2+\tfrac{1}{2}(\xi_1+a_1+a_2-1)^2.$$

The summation is readily carried out by means of the formula [18]

$$\sum_{k\in\mathbb{Z}}(-z)^k x^{k(k-1)/2}=E(z,x).$$

Thus we find that (4.14) is equal to

$$\frac{\varphi(-\sqrt{x})E(-x^{6a_1-2a_2+3/2},x^3)-\varphi(\sqrt{x})E(x^{6a_1-2a_2+3/2},x^3)}{2\varphi(x)^4}x^{7a_1^2-3a_1a_2+a_2^2-1/2}.$$

The expression can be further reduced thanks to the identity $E(zx^k,x)=(-z)^{-k}x^{-k(k-1)/2}E(z,x)$, $\forall k\in\mathbb{Z}$. Calculation of the denominator in (4.6*b*) (or (4.12)) goes in a parallel way. Below we present the final results for the LSP $P(a|\sigma_k)$ for $a=(1-2a_1-a_2)\Lambda_0+a_1\Lambda_1+a_2\Lambda_2$, $a_1, a_2\in\mathbb{Z}$, $k=0,1,2$:

$$q^{-1}P(a|\sigma_1)\pm qP(a|\sigma_0)=\begin{cases}\dfrac{2\varphi(\mp q^{12})E(\mp q^{36},q^{72})}{D_++D_-}q^{6\langle a,2a-\rho\rangle-1} & \text{if } a_2\equiv 0 \bmod 3\\ \pm\dfrac{2\varphi(\mp q^{12})E(\mp q^{12},q^{72})}{D_++D_-}q^{6\langle a,2a-\rho\rangle+3} & \text{otherwise}\end{cases} \tag{4.15a}$$

$$P(a|\sigma_2)=\begin{cases}\dfrac{E(-1,q^{72})}{D_0}q^{6\langle a,2a-\rho\rangle+10} & \text{if } a_2\equiv 0 \bmod 3\\ \dfrac{E(-q^{24},q^{72})}{D_0}q^{6\langle a,2a-\rho\rangle+2} & \text{otherwise}\end{cases} \tag{4.15b}$$

$$D_\pm=\varphi(\pm q^{12})E(\pm q^2,q^{24})E(\pm q^4,q^{24})E(\pm q^{10},q^{24}) \tag{4.15c}$$

$$D_0=E(-q^2,q^{24})E(-q^8,q^{24})E(-q^{10},q^{24}). \tag{4.15d}$$

The LSP (4.15) exhibit the behaviour

$$\lim_{q\to 0}P(a|\sigma_k)=\delta_{a\sigma_k^{(1)}} \tag{4.16a}$$

$$\lim_{q\to 1}P(a|\sigma_k)=0. \tag{4.16b}$$

This implies that our regime $0<q<1$ is in a low-temperature phase whose boundaries $q\to 0$ and $q\to 1$ correspond to the low-temperature limit and the critical point, respectively. When $q\to 1-0$, the LSP $P(a|\sigma_k)$ is of order $\ln q$. The difference $P(a|\sigma_1)-P(a|\sigma_0)$ vanishes as $(\ln q)\exp[7\pi^2/(288\ln q)]$.

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Appendix 1. Relevant results for a $B_n^{(1)}$ vertex model

Here we recapitulate the 1D sum results in [4] for the $B_n^{(1)}$ vertex model related to the vector representation of B_n. The affine Lie algebra $B_n^{(1)}$ is generated by the elements $\hat{H}_i, \hat{X}_i^{\pm}$ $(0 \leqslant i \leqslant n)$ and $\hat{d}$ obeying the commutation relations

$$\begin{aligned} &[\hat{H}_i, \hat{H}_j] = 0 \qquad [\hat{H}_i, \hat{X}_j^{\pm}] = \pm \hat{A}_{ji} \hat{X}_j^{\pm} \qquad [\hat{X}_i^+, \hat{X}_j^-] = \delta_{ij} \hat{H}_i \\ &[\hat{d}, \hat{X}_i^{\pm}] = \pm \delta_{0i} \hat{X}_i^{\pm} \qquad (\mathrm{ad}\, \hat{X}_i^{\pm})^{1-\hat{A}_{ji}} \hat{X}_j^{\pm} = 0 \quad (i \neq j). \end{aligned} \tag{A1.1}$$

Here the Cartan matrix has been chosen as $(\hat{A}_{ij})_{0 \leqslant i,j \leqslant n} = 2\delta_{ij} - \delta_{ij+1} - \delta_{ji+1}(1+\delta_{jn}) - \delta_{i2}\delta_{j0} - \delta_{i0}\delta_{j2} + \delta_{1i}\delta_{0j} + \delta_{0i}\delta_{1j}$. Let $\hat{\mathscr{H}}^* = \mathbb{C}\omega_0 \oplus \ldots \oplus \mathbb{C}\omega_n \oplus \mathbb{C}\hat{\delta}$ be the dual space of the Cartan subalgebra $\hat{\mathscr{H}}$. We introduce the orthonormal vectors e_i $(1 \leqslant i \leqslant n)$ in $\hat{\mathscr{H}}^*$ and express the fundamental weights ω_i and the simple roots $\hat{\alpha}_i$ as follows:

$$\begin{aligned} &\bar{\omega}_i = \omega_i - \langle \omega_i, \hat{\delta} \rangle \omega_0 \\ &\bar{\omega}_i = e_1 + \ldots + e_i \qquad 1 \leqslant i \leqslant n-1 \\ &\bar{\omega}_n = \tfrac{1}{2}(e_1 + \ldots + e_n) \\ &\hat{\alpha} \\ &\hat{\alpha}_n = e_n \\ &\hat{\delta} = \hat{\alpha}_0 + \hat{\alpha}_1 + 2(\hat{\alpha}_2 + \ldots + \hat{\alpha}_n) \\ &\langle \omega_i, \hat{\delta} \rangle = \begin{cases} 1 & \text{if } i = 0, 1, n \\ 2 & \text{otherwise.} \end{cases} \end{aligned} \tag{A1.2}$$

Assuming linearity and $\langle \hat{\delta}, \hat{\delta} \rangle = 0$, (A1.2) defines the inner product $\langle\ ,\ \rangle$ on the space $\hat{\mathscr{H}}^*$. We extend the index of the vector e_i to $-n \leqslant i \leqslant n$ by setting $e_{-i} = -e_i$. Note that $\{e_i | -n \leqslant i \leqslant n\}$ is the set of weights of the vector representation of B_n. We introduce an order $<$ in the index set $\{-n, -n+1, \ldots, n\}$ by $1 < 2 < \ldots < n < 0 < -n < -n+1 \ldots < -1$.

The 1D sum associated with the $B_n^{(1)}$ vertex model has the form

$$g_m(\xi, e_i; q) = \sum q^{\sum_{j=1}^m j \hat{H}(\xi^{(j)}, \xi^{(j+1)})}. \tag{A1.3a}$$

Here, $m \in \mathbb{Z}_{\geqslant 0}$, $-n \leqslant i \leqslant n$ and $\xi \in \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$. The outer sum is taken over $\xi^{(j)} \in \{e_k | -n \leqslant k \leqslant n\}$ $(1 \leqslant j \leqslant m+1)$ with the condition $\sum_{j=1}^m \xi^{(j)} = \xi$, $\xi^{(m+1)} = e_i$. The function $\hat{H}$ is defined by

$$\hat{H}(e_i, e_j) = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{otherwise} \end{cases}$$

with the exceptions

$$\hat{H}(e_1, e_{-1}) = -1 \qquad \hat{H}(e_0, e_0) = 0. \tag{A1.3b}$$

If we write $\xi = \xi_1 e_1 + \ldots + \xi_n e_n$, the following explicit formula is valid:

$$g_m(\xi, e_i; q) = \sum{}^* q^{\sum_{i=-n}^{n}[p_i(p_i-1)/2 + \hat{H}(e_i, e_i)p_i] - p_\lambda p_{-\lambda}} \begin{bmatrix} m \\ p \end{bmatrix}$$
$$\lambda = \begin{cases} n & \text{if } i \geq 0 \\ 1 & \text{otherwise.} \end{cases} \qquad \text{(A1.4}a\text{)}$$

The sum Σ^* is taken with respect to the variables $p_j \in \mathbb{Z}_{\geq 0}$, $-n \leq j \leq n$ under the constraint $\Sigma_{j=-n}^{n} p_j = m$ and $p_j - p_{-j} = \xi_j$ for $1 \leq j \leq n$. The symbol $[{m \atop p}]$ stands for q-multinomial coefficient [18]

$$\begin{bmatrix} m \\ p \end{bmatrix} = \frac{(q)_m}{\Pi_{j=-n}^{n} (q)_{p_j}} \qquad (q)_j = \prod_{k=1}^{j} (1 - q^k). \qquad \text{(A1.4}b\text{)}$$

The limit $m \to \infty$ of the 1D sum (A1.2) gives rise to level-1 string functions of the affine Lie algebra $B_n^{(1)}$. Let $L(\omega)$ be the irreducible highest-weight module with the highest weight ω. We set $L(\omega)_\mu = \{v \in L(\omega) | hv = \mu(h)v \text{ for } h \in \hat{\mathscr{H}}\}$. Define the sequence $\{\hat{\sigma}_\omega^{(j)}\}_{j \in \mathbb{Z}}$ of the level-1 integral weights by

$$\hat{\sigma}_\omega^{(j)} = \begin{cases} \omega_0 & \text{if } j \equiv k \bmod 2 \\ \omega_1 & \text{otherwise} \end{cases} \qquad \text{for } \omega = \omega_k \ (k = 0, 1) \qquad \text{(A1.5)}$$
$$= \omega_n \qquad \forall j \qquad \text{for } \omega = \omega_n.$$

Assuming that $\hat{a} \in \mathbb{Z}\omega_0 \oplus \ldots \oplus \mathbb{Z}\omega_n$, $\langle \hat{a}, \hat{\delta} \rangle = 1$ and $\omega \in \{\omega_0, \omega_1, \omega_n\}$, the main result for the 1D sum is stated as follows:

$$\lim_{m \to \infty} q^{-\psi_m(\omega)} g_m(\hat{\sigma}_\omega^{(m+1)} - \hat{a}, \hat{\sigma}_\omega^{(m+2)} - \hat{\sigma}_\omega^{(m+1)}; q) = \sum_i \dim L(\omega)_{\hat{a} - i\hat{\delta}} q^i \qquad \text{(A1.6}a\text{)}$$

$$\psi_m(\omega) = \sum_{j=1}^{m} j \hat{H}(\hat{\sigma}_\omega^{(j+1)} - \hat{\sigma}_\omega^{(j)}, \hat{\sigma}_\omega^{(j+2)} - \hat{\sigma}_\omega^{(j+1)}). \qquad \text{(A1.6}b\text{)}$$

The RHS of (A1.6a) is the $B_n^{(1)}$ string function [10] on the module $L(\omega)$. In fact, by taking the $m \to \infty$ limit in (A1.4), the following explict formulae are available, including (A1.6):

$$\lim_{m \to \infty, m \in 2\mathbb{Z}} q^{t_m} g_m(\xi, e_i; q)$$
$$= \begin{cases} q^{(1/2)|\xi - \omega_0|^2} \dfrac{\varphi(-\sqrt{q}) + (-1)^s \varphi(\sqrt{q})}{2\varphi(q)^{n+1}} \displaystyle\prod_{1 \leq j < i} (1 + q^{-\xi_j}) & 1 \leq i \leq n \\ q^{(1/2)|\xi - \omega_n|^2 - n/8} \dfrac{\varphi(q^2)}{\varphi(q)^{n+1}} & i = 0 \\ q^{(1/2)|\xi - \omega_n|^2 - n/8} \dfrac{\varphi(q^2)}{\varphi(q)^{n+1}} \displaystyle\prod_{-i < j \leq n} (1 + q^{\xi_j}) & -n \leq i < -1 \\ q^{(1/2)|\xi - \omega_1|^2 - 1/2} \dfrac{\varphi(-\sqrt{q}) + (-1)^s \varphi(\sqrt{q})}{2\varphi(q)^{n+1}} & i = -1 \end{cases} \qquad \text{(A1.7}a\text{)}$$

where t_m and s are given by

$$t_m = \begin{cases} \dfrac{m}{2} \delta_{-1i} & \text{if } i < 0 \\ \dfrac{m}{2} (\delta_{0i} - 1) & \text{otherwise} \end{cases} \qquad s = \sum_{j=1}^{n} \xi_j. \qquad \text{(A1.7}b\text{)}$$

Appendix 2. The embedding $G_2^{(1)} \subset B_3^{(1)}$

The embedding of the affine Lie algebra $G_2^{(1)}$ in $B_3^{(1)}$ is the key to characterising our 1D sum (4.7). Here, the generators of $G_2^{(1)}$ satisfying (A1.1), but with $(\hat{A}_{ij})_{0\leq i,j\leq 2} = 2\delta_{ij} - \delta_{ij+1} - \delta_{ji+1}(1+2\delta_{j2})$, will be denoted by $x_i^{\pm}$, h_i $(0\leq i\leq 2)$ and d. Let $\hat{\mathcal{H}}^*$ be as defined in appendix 1 with $n=3$, i.e. the dual space of the Cartan subalgebra for $B_3^{(1)}$. We define a $\mathbb{C}$-linear map $\phi: \hat{\mathcal{H}}^* \to \mathcal{H}^*$ by

$$\begin{aligned} &\phi(\omega_0) = \Lambda_0 \qquad \phi(\omega_1) = \Lambda_2 \\ &\phi(\omega_2) = \Lambda_1 \qquad \phi(\omega_3) = \Lambda_2 \qquad \phi(\hat{\delta}) = \delta. \end{aligned} \tag{A2.1}$$

Thus, in particular, we have the following by using (2.1), (2.2) and (A1.2):

$$\begin{aligned} &\phi(e_{\pm 1}) = \varepsilon_{\mp 3} \qquad \phi(e_{\pm 2}) = \varepsilon_{\mp 2} \\ &\phi(e_{\pm 3}) = \varepsilon_{\mp 1} \qquad \phi(e_0) = \varepsilon_0. \end{aligned} \tag{A2.2}$$

This ϕ induces the map $\phi^*: G_2^{(1)} \to B_3^{(1)}$ as follows:

$$\begin{aligned} &\phi^*(x_0^{\pm}) = \hat{X}_0^{\pm} \qquad \phi^*(h_0) = \hat{H}_0 \\ &\phi^*(x_1^{\mp}) = \hat{X}_2^{\pm} \qquad \phi^*(h_1) = \hat{H}_2 \\ &\phi^*(x_2^{\mp}) = \hat{X}_1^{\mp} + \hat{X}_3^{\pm} \qquad \phi^*(h_2) = \hat{H}_1 + \hat{H}_3 \\ &\phi^*(d) = \hat{d}. \end{aligned} \tag{A2.3}$$

The consistency of the commutation relations under (A2.3) can be directly checked. Thus ϕ^* defined as above gives the embedding of the affine Lie algebra $G_2^{(1)} \hookrightarrow B_3^{(1)}$.

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